Very Preliminary

Mathematical Analytics of Lewisian Dual-Economy Model:
How Capital Accumulation and Labor Migration Promote Development

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Summary

This study explores a theoretical model for economic development that was presented by Lewis (1954) and develops its mathematical representation in particular. A dual-economy model will be considered based on Lewis (1954), consisting of two sectors, such as the traditional subsistence sector and the modern capitalist sector. This study reveals the roles of the capital accumulation and the labor migration in Lewisian model from the viewpoint of mathematical aspect. According to this analysis following points are important for economic take-off in developing countries:
(1) promoting the capital accumulation, in particular introduction of high productivity capital; and
(2) encouraging the smooth labor migration from the subsistence sector to the capitalist sector with modest population growth.

1. Introduction

The latest update of *World Economic Outlook* of the IMF expects the global economy to going back to a steady growth path in some countries around 2021-22 after experiencing the negative economic impact of COVID-19, which is wearing off. Figure 1 depict the latest world Economic Outlook Update Growth Projections. There are many factors required for sustainable growth, among those, one of the most important for economic development in emerging and developing countries is capital accumulation.[1]

Table 1: Latest World Economic Outlook Update Growth Projections

		Projections	
(real GDP, annual percent change)	2020	2021	2022
World Output	-3.2	6.0	4.9
Advanced Economies	-4.6	5.6	4.4
United States	-3.5	7.0	4.9
Euro Area	-6.5	4.6	4.3
Germany	-6.5	4.6	4.3
France	-8.0	5.8	4.2
Italy	-8.9	4.9	4.2
Spain	-10.8	6.2	5.8
Japan	-4.7	2.8	3.0
United Kingdom	-9.8	7.0	4.8
Canada	-5.3	6.3	4.5
Other Advanced Economies	-2.0	4.9	3.6
Emerging Market and Developing Economies	-2.1	6.3	5.2
Emerging and Developing Asia	-0.9	7.5	6.4
China	2.3	8.1	5.7
India	-7.5	9.5	8.5
ASEAN 5	-3.4	4.3	6.3
Emerging and Developing Europe	-2.0	4.9	3.6
Russia	-3.0	4.4	3.1
Latin America and the Caribbean	-7.0	5.8	3.8
Brazil	-4.1	5.3	1.9
Mexico	-8.3	6.3	4.2
Middle East and Central Asia	-2.6	4.0	3.7
Saudi Arabia	-4.1	2.4	4.8
Sub-Sahara Africa	-1.8	3.4	4.1
Nigeria	-1.8	2.5	2.6
South Africa	-7.0	4.0	2.2
Memorandum			
Emerging Market and Middle-Income Economies	-2.3	6.5	5.2
Low-Income Developing Countries	0.2	3.9	5.5

note: related to regional divisions, see IMF (2021).
source: IMF (2021)

This study intends to explore a theoretical model for economic development and to find some implications to both emerging and developing countries, including some Southeast Asian countries. Lewis (1954) originally presented this type of dual-economy model and

a lot of economists have been developing it including Leamer (1987)[2], Banerjee and Newman (1996) and Freeman and Lindauer (1999). Harris and Todaro (1970) include a politically determined minimum urban wage rate to dual-economy model while Emerson (2002) adds an element of corruption in a model. Solow (1956) and Goodfriend and McDermott (1995) contribute to theoretical aspect to a great deal.

This paper consists of five chapters including this introduction; the following second chapter presents some assumptions for the model, including sector structure, etc.; the third chapter focuses on a theoretical framework of the model that are production functions, determination of wage and income, capital accumulation function, demand functions, and so on: the fourth chapter mathematically analyzes the model, using mainly differentiation; and the final and fifth chapter briefly concludes the discussion in the study to derive some policy implications for development economics in general and policies for Official Development Assistance (ODA). Moreover, this study shows that this dual-economy model is also applicable for one-time Japanese postwar high growth period analyzed by Yoshioka and Kawasaki (2016).

2. Basic Features and Assumptions

Some basic features, including assumptions of Lewisian dual sector model in this study are as follows:
(1) The economy consists of two sectors such as the capitalist and the subsistence sectors.
(2) At the capitalist sector, wage rate is determined by marginal productivity of labor as usually assumed in mainstream economics.
(3) At the subsistence sector, wage rate is equal to subsistence level while marginal productivity of labor is close to zero, which means that labor loses its scarcity.[3]
(4) All of capital stock exists only at the capitalist sector while none utilized at the subsistence sector.
(5) The capital accumulation or investment originates from the benefit of capital while wages at both sectors are consumed totally.
(6) The capitalist sector goods are utilized for both consumption and investment while the subsistence sector goods are only for consumption.
(7) Large and disguised unemployment exists at the subsistence sector, which could be a nearly infinite resource for unskilled labor supply for the capitalist sector.[4]
(8) Prices of the subsistence sector goods are utilized as a measuring standard for the wage rate while the capitalist sector goods play the same role for subsistence sector goods.
(9) The model itself is not dynamic but based on a comparative-static basis.
(10) A closed economy is assumed for simplification of the model.[5]

3. Theoretical Framework

Lewisian dual-economy model then could be mathematically represented as following set of equations:
(EQ1)   production functions[6]
$$Y = Y_C + Y_S$$
$$L = L_C + L_S$$
$$Y_C = f\ K, L_C)$$

$$Y_S = g\ L_S)^7$$

(EQ2) wage rates[8] and aggregated wage incomes

$$w_C = \frac{1}{p} \cdot \frac{\partial f}{\partial L_C}$$

$$w_S > \frac{\partial g}{\partial L_S} \cong 0$$

$$w_C > w_S$$

$$W_C = w_C L_C$$

$$W_S = w_S L_S$$

$$W = W_C + W_S$$

(EQ3) capital accumulation[9]

$$\frac{dK}{dt} = rK - \mu K$$

or $$K = \int_{-\infty}^{0} rK - \mu K)dt$$

(EQ4) demand functions

$$D = D_C + D_S$$

$$D_C = \varphi\ p, r, W_C, W_S, Y_S - W_S)$$

$$D_S = \psi\ p, W_C, W_S, Y_S - W_S)$$

(EQ5) coincidence of demand and supply

$$Y = D$$

$$Y_C = D_C$$

$$Y_S = D_S$$

where Y_i output ($i=C$, S or none)
        f, g production functions
        K capital stock (only at capitalist sector)
        L_i labor ($i=C$, S or none)
        w_i wage rate ($i=C$ or S)
        W_i wage income ($i=C$, S or none)
        p price level of subsistence goods measured by capitalist goods
        r profit rate of capital at capitalist sector or marginal
                     productivity of capital or $r = \frac{\partial f}{\partial K}$
        μ depreciation rate of capital
        t time
        D_i demand ($i=C$, S or none)
        φ, ψ demand functions

4. Mathematical Analysis

The third equation of (EQ4) implies that the demand for subsistence goods does not depend on capital profit at capitalist sector but on following four factors:

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(1) price level (i.e., p);
(2) labor income at the capitalist sector (i.e., W_C);
(3) labor income at the subsistence sector (i.e., W_S); and
(4) non-wage income at subsistence sector (i.e., Y_S-W_S).

Based on the equations of (EQ1), (EQ4) and (EQ5), the following equations of (EQ6) apparently hold:
(EQ6) coincidence of demand and supply of subsistence goods
$$g\ L_S) = \psi\ p, W_C, W_S, Y_S - W_S)$$
or $$g\ L_S) = \lambda\ p, W_C, W_S)$$
  where $$\lambda\ p, W_C, W_S) \equiv \psi\ p, W_C, W_S, Y_S - W_S)$$

The latter equation of (EQ6) can be solved as a function of price level of subsistence goods measured by capitalist goods as follows:
(EQ7) prices of subsistence goods measured by capitalist goods or price function
$$p = \pi\ w_C, w_S, L_C, L_S)$$
  where π price function

Supposed that the labor supply is constant in a short run, the labor migration from the subsistence sector to the capitalist sector brings following four results:
(1) decrease of subsistence sector goods output;
(2) increase of wage income at the capitalist sector;
(3) decrease of wage income at the subsistence sector; and
(4) increase of total wage income.

These results are mathematically represented as follows:
(EQ8) effect of labor migration from the subsistence sector to the capitalist sector

$$\frac{\partial g}{\partial L_C} < 0 \qquad \text{since} \qquad \frac{\partial g}{\partial L_S} < 0 \ \text{ and } \ \frac{\partial L_S}{\partial L_C} < 0$$

$$\frac{\partial W_C}{\partial L_C} > 0$$

$$\frac{\partial W_s}{\partial L_C} > 0$$

$$\frac{\partial W}{\partial L_C} > 0 \qquad \text{since} \qquad w_C > w_S$$

The first result of decreasing supply of subsistence sector goods might also bring a hike of its prices if its decrease of demand exceeds its decrease of supply. [10] It's because the elasticity for subsistence sector goods against both income and prices is rather small. It, however, depends on the amount of labor migration since marginal labor productivity at the subsistence sector is assumed as close to zero. Following Figure 2 reports a conseputual image of this effect of labor migration on its marginal productivity.

Figure 2: Labor and Marginal Productivity/Output at Subsistence Sector
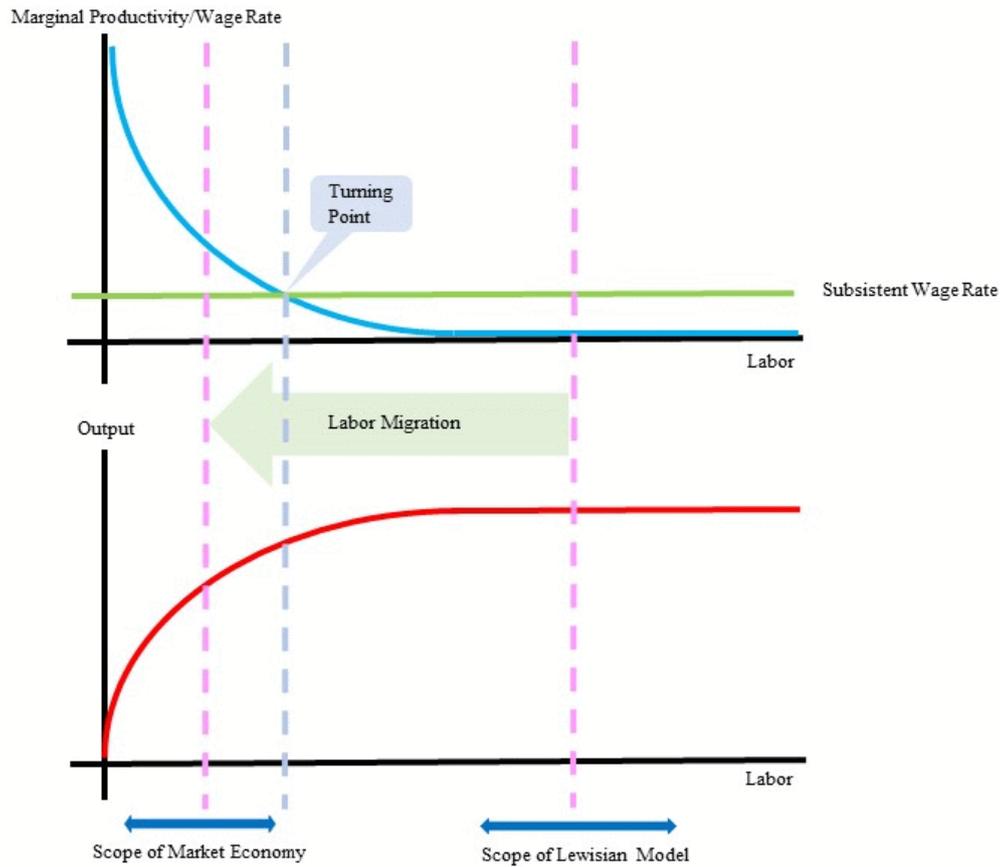
Source: author based on Yoshioka and Kawasaki (2016) Figure A1-4, p.78

On the other hand, also suppose that the capital stock is constant in a short run, price level of subsistence goods measured by capitalist goods (i.e., p) and labor at capitalist sector (i.e., L_C) are decided under the condition of the first equation of (EQ2) and the price function of (EQ7). At the above equations, other factors than these two variables such as p and L_C are not variables but intercept, which implies that they are decided according to functions of p, L_C and K. Differentiating the first equation of (EQ2) and the equation of (EQ7), we then obtain following equations:

(EQ9)   results of differentiation

$$\frac{\partial^2 f}{\frac{\partial L_C}{\partial K}} = \frac{\partial^2}{\partial L_C{}^2} \cdot \frac{dL}{dK} - \frac{\partial f}{\partial L} \cdot \frac{dp}{dK}$$

$$\frac{\partial \pi}{\partial L_C} \cdot \frac{dL_C}{dK} - \frac{dp}{dK} = 0$$

And solving above two equations of (EQ9) on $\frac{dp}{dK}$, we then obtain the results that capital accumulation has a positive differential coefficient on subsistence sector goods prices under the condition of sizable labor migration[11] as follows:

(EQ10) effect of capital accumulation on subsistence goods price

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$$\frac{dp}{dK} = \frac{-\frac{\partial \pi}{\partial L_C} \cdot \frac{\partial^2 f}{\partial L_C \partial K}}{\frac{\partial^2 f}{\partial L_C{}^2} - \frac{\partial f}{\partial L_C} \cdot \frac{\partial \pi}{\partial L_C}} > 0$$

since $\dfrac{\partial^2 f}{\partial L_C \partial K} > 0$, $\dfrac{\partial^2 f}{\partial L_C{}^2} > 0$, and $\dfrac{\partial f}{\partial L_C} > 0$

And also solving them on $\dfrac{dL_C}{dK}$, we then obtain the results that capital accumulation, of course, has a positive differential coefficient on labor at the capitalist sector as follows:

(EQ11) effect of capital accumulation on labor at capitalist sector

$$\frac{dL_C}{dK} = \frac{-\frac{\partial^2 f}{\partial L_C \partial K}}{\frac{\partial^2 f}{\partial L_C{}^2} - \frac{\partial f}{\partial L_C} \cdot \frac{\partial \pi}{\partial L_C}} > 0$$

since $\dfrac{\partial^2 f}{\partial L_C \partial K} > 0$, $\dfrac{\partial^2 f}{\partial L_C{}^2} > 0$, and $\dfrac{\partial f}{\partial L_C} > 0$

These mathematical equations naturally represent that capital accumulation at capitalist sector leads to following results:
(1) hike of the subsistence sector goods prices; and
(2) increase of labor at the capitalist sector.

In addition, focusing on the first equation of (EQ3), we then find out follows:

(EQ12) investment or capital accumulation

$$\frac{dK}{dt} = rK - \mu K = \frac{\partial f}{\partial K} K - \mu K$$

or $\quad \dfrac{\frac{dK}{dt}}{K} = \dfrac{\partial f}{\partial K} - \mu$

The left hand of the second equation of (EQ12) reports growth rate of capital accumulation. When the benefit rate or the marginal productivity of capital exceeds the depreciation rate the capitalist sector grows naturally. However, since it is also usually assumed that $\dfrac{\partial^2 f}{\partial K^2} < 0$, capital accumulation stops after marginal productivity of capital falling below the depreciation rate. The biggest problem is when it stops, after labor has migrated enough to pull up wage rate at the subsistence sector to exceed the subsistence level or before?[12]

Another important viewpoint is related to labor, or population that strongly correlates labor. The higher the growth rate of population is, the more labor migration from the subsistence sector to capitalist sector is required for pulling up marginal productivity of labor at the subsistence sector or 'take-off.' From this perspective, population growth must be within moderate limits, and a population explosion is not desirable for economic development.

5. Conclusion

The theoretical dual-economy model for economic development analyzed in this study is very preliminary so far. A further investigation would be required. One of the most required viewpoints for expansion of this model is to include foreign sector that might provide net import (or aid) and foreign direct investment that contributes to capital accumulation to a great extent. Apart from this, it is appropriate to point out that following two respects seem important for economic development or 'take-off.'

(1) to promote capital accumulation, in particular introduction of high productivity capital; and

(2) to encourage smooth labor migration from the subsistence sector to the capitalist sector with modest population growth.

Figure 3: Conditional Correlation Between Growth and Total Aid, 1960–2000

coef = -.06308045, (robust) se = .02796237, t = -2.26
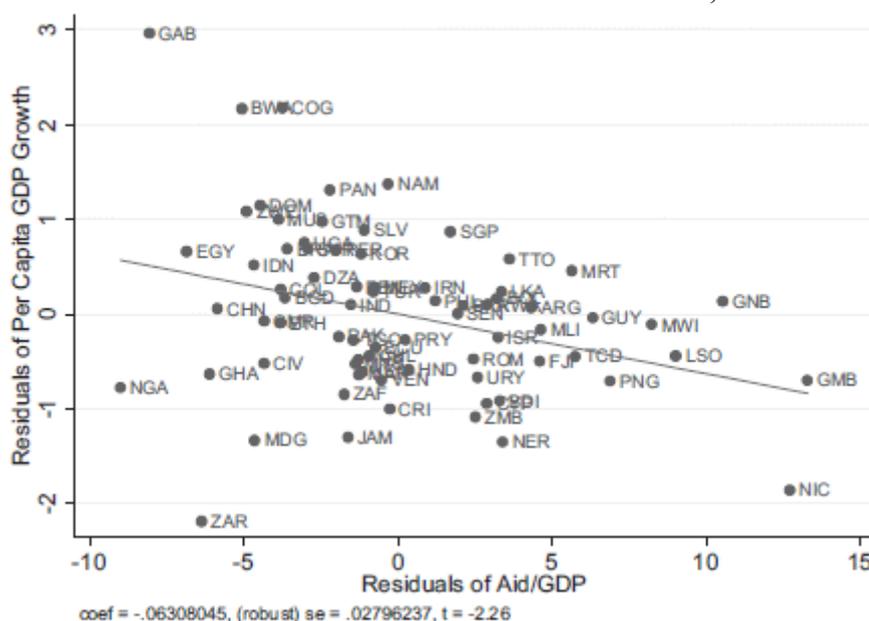
source: Rajan and Subramanian (2008) Figure 1, p.647

Some literature on development economics dealing with the Official Development Assistance (ODA) suggests that ODA has little effect on growth.[13] For example, Figure 3 quoted from Rajan and Subramanian (2008) insists that they found somehow negative correlation between growth and total aid. However, we can also derive some implication for ODA policies from to developing countries from the perspective of Lewisian dual-economy model as follows:

(1) ODA must help promoting both capital accumulation and labor migration in developing countries; and

(2) ODA is also required to prevent decrease of the subsistence sector output when sizable labor migration[14] is going on or after it has finished while ODA for subsistence sector before it might discourage autonomous labor migration from the subsistence sector to the capitalist sector.

Figure 4: GDP Growth Rate in postwar Japan
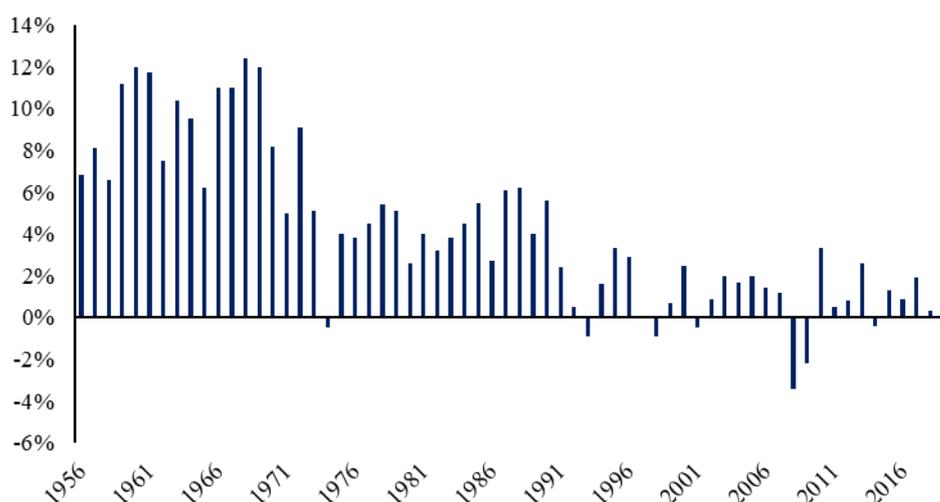
source: SNA statistics, Cabinet Office

Finally, after the World War II, Japan experienced high-growth period in 1950s and 60s. entering to so-called stable growth period after the oil crises, but Japan's growth rate declined to a considerable extent as Figure 4 depicts. And in the 21st century, Japan's growth seems to have disappeared. Related to this Japan's growth, Ohkita (2010) examines the reasons why Japan's rapid growth in the 1950s and 1960s was a one-time event. As well as Japan, some developed countries also enjoyed high growth period after the war. In the case of France, Fourastié (1979) describes it very clearly. When we adopt this Lewisian dual-economy model for early development stage, this question of Ohkita (2010) could be easily answered. Many economists would agree that the large-scale labor migration, as assumed in the Lewisian model, occurs only once in a long period of economic history.

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[1] Mankiw et al. (1992) and Rappaport (2006) strongly insist of this point.

[2] Leamer (1987) doesn't take dual-economy model but explores three-factor model.

[3] Jorgenson (1961) suggests that this assumption is not necessarily required for development analysis.

[4] Sen (1966) includes some examples without labor surplus.

[5] FDI (foreign direct investment) and/or ODA (official development assistance) could be included as a resource of investment. This paper, however, excludes them for simplification.

[6] The suffixes C and S denote the capitalist and subsistence sectors, respectively (same below).

[7] The output at the subsistence sector depends only on labor.

[8] The wage rate at both sectors is measured by the subsistence sector goods prices according to the assumption (7).

[9] Investment or capital accumulation is carried out only in the capitalist sector.

[10] This implies the possibility that $\frac{\partial \pi}{\partial L_C} > 0$ could be satisfied.

[11] Here, "sizable labor migration" means that it satisfies $\frac{\partial \pi}{\partial L_C} > 0$.

[12] Of course, when labor migration pulls up wage at the subsistence sector up to its marginal

productivity, this sector is no longer subsistent. Due to assumptions and definitions of Lewisian model, this economy has already succeeded in 'take-off.' Moreover, from a historical viewpoint, no economy has so far reached at this stage that capital accumulation ceases.

[13] Other than Rajan and Subramanian (2008), Doucouliagos and Paldam (2009) also have the same opinion.

[14] Here, I use "sizable labor migration" such that brings decrease of subsistence sector goods output. See also endnote10.